APPENDIX

Transformations

Contents

0. Foreword
1. Transformation Laws for Cartesian Tensors
2. Transformation of Displacement and Force Vectors
3. Transformation of Strain and Stress Matrices
4. Transformation of Constitutive Matrices
5. References
0. Foreword

The one-dimensional examples presented in Chapter 5 have involved axial elements aligned parallel with the global $x_1$-axis. To be useful in general applications, these formulations must be generalized to elements oriented at any angle with respect to the global coordinate system. This requires the use of suitable transformations between vector and matrix quantities in local (element) coordinates, and those in global coordinates.

The present development begins with the more general notion of Cartesian tensors. Then, with an eye towards eventual numerical implementation, the discussion is specialized to vectors and matrices typically associated with finite element analyses.

1. Transformation Laws for Cartesian Tensors

In the sequel, the local (element) coordinates shall be represented as primed, while the global coordinates shall be unprimed. Denote the cosine of the angle between the i-th primed and the j-th unprimed coordinate axes by

$$ r_{ij} = \cos(\chi'_i, \chi_j) \quad (1) $$

where the standard laws for indicial notation apply. In light of the orthonormality of direction cosines, it follows that

$$ r_{ij} r_{ik} = \delta_{jk} \quad (2) $$

where $\delta_{ij}$ represents the Kronecker delta.

Using standard relations from vector algebra, a unit vector in the $x'_i$ direction is related to the unit vectors associated with the unprimed coordinates by

$$ \hat{e}'_i = r_{i1} \hat{e}_1 + r_{i2} \hat{e}_2 + r_{i3} \hat{e}_3 = r_{ij} \hat{e}_j \quad (3) $$

It follows that in general

$$ \hat{e}'_i = r_{ij} \hat{e}_j \quad (4) $$

Consider next the first-order Cartesian tensor $v$. With respect to the unprimed and primed coordinate system, $v$ is written as

$$ v = v_j \hat{e}_j \quad (5) $$
\[ \mathbf{v} = \mathbf{v}' \mathbf{\hat{e}}'_i \]  
(6)

implying that
\[ v_j \mathbf{\hat{e}}_j = v'_i \mathbf{\hat{e}}'_i \]  
(7)

Substituting equation (4) into equation (7) finally gives the transformation law for first-order Cartesian tensors
\[ v_j = r_{ij} v'_i \]  
(8)

or
\[ v'_i = r_{ij} v_j \]  
(9)

For the case of a second-order Cartesian tensor \( \mathbf{T} \), the above results generalize to
\[ T'_{ij} = r_{im} r_{jn} T_{mn} \]  
(10)

The inverse relation is given by
\[ T_{ij} = r_{mi} r_{nj} T'_{mn} \]  
(11)

Finally, the fourth-order Cartesian tensor \( \mathbf{C} \) transforms according to the following law
\[ C'_{ijkl} = r_{ip} r_{jq} r_{kr} r_{ls} C_{pqrs} \]  
(12)

with the inverse relation
\[ C_{ijkl} = r_{pl} r_{ql} r_{rl} r_{sl} C'_{pqrs} \]  
(13)

To facilitate their eventual adoption into numerical schemes, first order Cartesian tensors shall, in the sequel, be represented by \((3 \times 1)\) vectors. Furthermore, second-order tensors shall be represented by \((3 \times 3)\) matrices. Due to symmetries associated with the strain and stress tensors, these \((3 \times 3)\) matrices shall further be reduced to \((6 \times 1)\) vectors. Finally, owing to the aforementioned symmetries, fourth-order tensors shall be represented by \((6 \times 6)\) matrices.

Since the transformation tensor is second-order, it may be represented by the \((3 \times 3)\) matrix \( \mathbf{R} \), where
\[
\mathbf{R} = \begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
\]  
(14)

To gain some insight into \( \mathbf{R} \), consider each of its three columns as vectors. These vectors have the following properties: (1) each is a unit vector and, (2) each is orthogonal to the others (i.e., their dot products are zero). The above properties also hold for the rows of \( \mathbf{R} \). Since its
columns are mutually orthogonal unit vectors, \( R \) is said to be an *orthogonal* matrix. From the orthonormality relation described by equation (2), it follows that

\[
RR^T = I \quad \Rightarrow \quad R^T = R^{-1} \quad (15)
\]

**Example 1.** Specific Transformation Matrices

Consider the special case of a rotation about the \( x_1 \)-axis.

**Figure E.1.1. Rotation About \( x_1 \)-Axis**

In this case \( r_{11} = 1, r_{12} = r_{21} = r_{13} = r_{31} = 0 \). The associated transformation matrix is thus

\[
R_1 = \begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \alpha_1 & \sin \alpha_1 \\
0 & -\sin \alpha_1 & \cos \alpha_1
\end{bmatrix} \quad (1)
\]

Considering each of the columns or rows of \( R_1 \) as vectors, it is evident that each is a unit vector, and that each is orthogonal to the other two.

**Figure E.1.2. Rotation About \( x_2 \)-Axis**

By analogy to the above results, it follows that the transformation matrix associated with rotation about the \( x_2 \)-axis (Figure E.1.2) will be given by
\[ R_2 = \begin{bmatrix} \cos \alpha_2 & 0 & -\sin \alpha_2 \\ 0 & 1 & 0 \\ \sin \alpha_2 & 0 & \cos \alpha_2 \end{bmatrix} \] 

Finally, for rotation about the \( x_3 \)-axes (Figure E.1.3), the transformation matrix is

\[ R_3 = \begin{bmatrix} \cos \alpha_3 & \sin \alpha_3 & 0 \\ -\sin \alpha_3 & \cos \alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \] 

**Figure E.1.3. Rotation About \( x_3 \)-Axis**

The total effect upon a vector of all three of the above rotations is equal to their product

\[ R_3 R_2 R_1 = \begin{bmatrix} \cos \alpha_3 & \sin \alpha_3 & 0 \\ -\sin \alpha_3 & \cos \alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \alpha_2 & 0 & -\sin \alpha_2 \\ 0 & 1 & 0 \\ \sin \alpha_2 & 0 & \cos \alpha_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha_1 & \sin \alpha_1 \\ 0 & -\sin \alpha_1 & \cos \alpha_1 \end{bmatrix} \]

\[ = \begin{bmatrix} \cos \alpha_2 \cos \alpha_3 & \cos \alpha_1 \sin \alpha_3 + \sin \alpha_1 \sin \alpha_2 \cos \alpha_3 & \sin \alpha_1 \sin \alpha_3 - \cos \alpha_1 \sin \alpha_2 \cos \alpha_3 \\ -\cos \alpha_2 \sin \alpha_3 & \cos \alpha_1 \cos \alpha_3 - \sin \alpha_1 \sin \alpha_2 \sin \alpha_3 & \cos \alpha_1 \sin \alpha_3 + \sin \alpha_1 \cos \alpha_2 \sin \alpha_3 \\ \sin \alpha_2 & -\sin \alpha_1 \cos \alpha_2 & \cos \alpha_1 \cos \alpha_2 \end{bmatrix} \]

As evident from equation (4), the effects of such rotations are *not* commutative.
2. Transformation of Displacement and Force Vectors

Since displacements and forces are vector quantities, it follows that their transformation to different coordinate systems is governed by equations (8) or (9).

Example 2. Transformation of Forces in Three-Dimensions

Consider a general force vector in the primed coordinate system

\[ \mathbf{f}' = f_{x_1}' \hat{\mathbf{e}}_{1}' + f_{x_2}' \hat{\mathbf{e}}_{2}' + f_{x_3}' \hat{\mathbf{e}}_{3}' \]  

(1)

Using the properties of the scalar (“dot”) product, the scalar component of the force along the unprimed \(x_1\)-axis is

\[ f_{x_1} = \mathbf{f}' \cdot \hat{\mathbf{e}}_1 = f_{x_1}' \hat{\mathbf{e}}_{1}' \cdot \hat{\mathbf{e}}_1 + f_{x_2}' \hat{\mathbf{e}}_{2}' \cdot \hat{\mathbf{e}}_1 + f_{x_3}' \hat{\mathbf{e}}_{3}' \cdot \hat{\mathbf{e}}_1 \]  

(2a)

\[ = f_{x_1}'r_{11} + f_{x_2}'r_{21} + f_{x_3}'r_{31} \]  

(2b)

Performing similar computations for the other two components gives, in matrix form

\[
\begin{bmatrix}
  f_{x_1} \\
  f_{x_2} \\
  f_{x_3}
\end{bmatrix} =
\begin{bmatrix}
  r_{11} & r_{21} & r_{31} \\
  r_{12} & r_{22} & r_{32} \\
  r_{13} & r_{23} & r_{33}
\end{bmatrix}
\begin{bmatrix}
  f_{x_1}' \\
  f_{x_2}' \\
  f_{x_3}'
\end{bmatrix}
\]  

(3a)

or

\[ \mathbf{f} = \mathbf{R}^T \mathbf{f}' \]  

(3b)

Equations (3) are seen to be nothing more than matrix representations of equation (8). Since \(\mathbf{R}\) is orthogonal, it follows that the inverse force transformation shall be given by

\[ \mathbf{f}' = \mathbf{R} \mathbf{f} \]  

(4)

The above results also directly apply to the transformation of displacement vectors. For example, consider the special case where the local and global \(x_3\)-axes are coincident, and a rotation about these axes is performed; i.e.,
In this case, \( r_{13} = r_{31} = r_{23} = r_{32} = 0 \), and \( r_{33} = 1 \) (see Example E.2.1). For convenience, the angle of rotation about the \( x_3 \)-axis is denoted by \( \theta_3 = \alpha \). As a result, the transformation law for displacements becomes

\[
\begin{bmatrix}
u'_1 \\
u'_2 \\
u'_3 \\
\end{bmatrix} =
\begin{bmatrix}
 r_{11} & r_{12} & 0 \\
r_{21} & r_{22} & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{bmatrix}
= \begin{bmatrix}
 \cos \alpha & \cos \beta & 0 \\
 \cos(\pi/2 + \alpha) & \cos \alpha & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{bmatrix}
= \begin{bmatrix}
 \cos \alpha & \sin \alpha & 0 \\
-\sin \alpha & \cos \alpha & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{bmatrix}
\]

\[ (5) \]
Example 3. Transformations Associated with Bar Elements.

Consider the bar element developed in Chapters 5 and 6, oriented in the general manner shown below

Associated with each of the approximate nodal displacement components are corresponding force components. In this case, the force components along the $x'_2$ and $x'_3$ axes are zero. As such, the general force transformation for node 1 reduces to

\begin{align*}
\mathbf{f}_1^{(1)} &= \mathbf{f}_1^{(1)} \cdot \hat{\mathbf{e}}_1 = \mathbf{f}_1^{(1)} \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = r_{11} \mathbf{f}_1^{(1)} \\
\mathbf{f}_2^{(1)} &= \mathbf{f}_1^{(1)} \cdot \hat{\mathbf{e}}_2 = \mathbf{f}_1^{(1)} \hat{\mathbf{e}}_2 \cdot \hat{\mathbf{e}}_2 = r_{12} \mathbf{f}_1^{(1)} \\
\mathbf{f}_3^{(1)} &= \mathbf{f}_1^{(1)} \cdot \hat{\mathbf{e}}_3 = \mathbf{f}_1^{(1)} \hat{\mathbf{e}}_3 \cdot \hat{\mathbf{e}}_3 = r_{13} \mathbf{f}_1^{(1)}
\end{align*}

or, in matrix form

\[
\begin{bmatrix}
\mathbf{f}_1^{(1)} \\
\mathbf{f}_2^{(1)} \\
\mathbf{f}_3^{(1)}
\end{bmatrix} =
\begin{bmatrix}
r_{11} & 0 & 0 \\
r_{12} & 1 & 0 \\
r_{13} & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\mathbf{f}_1^{(1)} \\
\mathbf{f}_2^{(1)} \\
\mathbf{f}_3^{(1)}
\end{bmatrix} = \mathbf{R}^T \mathbf{f}^{(1)}
\]

Extending the above relation to the entire element, and deleting unnecessary columns in $\mathbf{R}^T$ and rows in $\mathbf{f}^{(1)}$ leads to
Appendix : Transformations

\[
\begin{pmatrix}
  f_1^{(1)} \\
  f_2^{(1)} \\
  f_3^{(1)} \\
  f_1^{(2)} \\
  f_2^{(2)} \\
  f_3^{(2)}
\end{pmatrix}
= 
\begin{bmatrix}
  r_{11} & 0 \\
  r_{12} & 0 \\
  r_{13} & 0 \\
  0 & r_{11} \\
  0 & r_{12} \\
  0 & r_{13}
\end{bmatrix}
\begin{pmatrix}
  f_1'^{(1)} \\
  f_1'^{(2)}
\end{pmatrix}
\]

which is the transformation relation for axial force vectors in space.

3. Transformation of Strain and Stress Matrices

Since strain and stress represent second-order tensors, they transform according to equations (10) and (11). Thus, in the case of the strain tensor

\[
\varepsilon_{ij}' = r_{ip} r_{jq} \varepsilon_{pq}
\]

or upon expansion

\[
\begin{bmatrix}
  \varepsilon_{11}' & \varepsilon_{12}' & \varepsilon_{13}' \\
  \varepsilon_{21}' & \varepsilon_{22}' & \varepsilon_{23}' \\
  \varepsilon_{31}' & \varepsilon_{32}' & \varepsilon_{33}'
\end{bmatrix}
= 
\begin{bmatrix}
  r_{11} & r_{12} & r_{13} \\
  r_{21} & r_{22} & r_{23} \\
  r_{31} & r_{32} & r_{33}
\end{bmatrix}
\begin{bmatrix}
  \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\
  \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\
  \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33}
\end{bmatrix}
\begin{bmatrix}
  r_{11} & r_{21} & r_{31} \\
  r_{12} & r_{22} & r_{32} \\
  r_{13} & r_{23} & r_{33}
\end{bmatrix}
\]

The stress tensor transforms in an identical manner; that is,

\[
\sigma_{ij}' = r_{ip} r_{jq} \sigma_{pq}
\]

To gain better insight into the transformation of stresses, consider a two-dimensional state of stress. The general transformation law then reduces to (see also Figure E.2.1)

\[
\begin{bmatrix}
\sigma'_{11} & \sigma'_{12} & 0 \\
\sigma'_{21} & \sigma'_{22} & 0 \\
0 & 0 & \sigma'_{33}
\end{bmatrix}
= \begin{bmatrix}
r_{11} & r_{12} & 0 \\
r_{21} & r_{22} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{21} & \sigma_{22} & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
r_{11} & r_{12} & 0 \\
r_{21} & r_{22} & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

(1)

Evaluating the right hand side gives

\[
\sigma'_{11} = r_{11}[r_{11}\sigma_{11} + r_{12}\sigma_{12}] + r_{12}[r_{11}\sigma_{21} + r_{12}\sigma_{22}] 
\]

(2)

\[
\sigma'_{22} = r_{22}[r_{21}\sigma_{11} + r_{22}\sigma_{12}] + r_{21}[r_{21}\sigma_{21} + r_{22}\sigma_{22}] 
\]

(3)

\[
\sigma'_{12} = \sigma'_{21} = r_{11}[r_{21}\sigma_{11} + r_{22}\sigma_{12}] + r_{12}[r_{21}\sigma_{21} + r_{22}\sigma_{22}] 
\]

(4)

\[
\sigma'_{33} = \sigma_{33} 
\]

(5)

Noting from Figure E.2.1 that \( r_{11} = \cos \alpha, \ r_{12} = \cos \beta = \sin \alpha, \ r_{21} = -\sin \alpha, \) and \( r_{22} = \cos \alpha, \) it follows that

\[
\sigma'_{11} = (\cos \alpha)^2 \sigma_{11} + 2(\cos \alpha)(\sin \alpha)\sigma_{12} + (\sin \alpha)^2 \sigma_{22} 
\]

(6a)

\[
= \frac{(1 + \cos 2\alpha)}{2} \sigma_{11} + (\sin 2\alpha)\sigma_{12} + \frac{(1 - \cos 2\alpha)}{2} \sigma_{22} 
\]

(6b)

\[
= \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\alpha + (\sin 2\alpha)\sigma_{12} 
\]

(6c)

\[
\sigma'_{22} = (\sin \alpha)^2 \sigma_{11} - 2(\cos \alpha)(\sin \alpha)\sigma_{12} + (\cos \alpha)^2 \sigma_{22} 
\]

(7a)

\[
= \frac{(1 - \cos 2\alpha)}{2} \sigma_{11} - (\sin 2\alpha)\sigma_{12} + \frac{(1 + \cos 2\alpha)}{2} \sigma_{22} 
\]

(7b)

\[
= \frac{\sigma_{11} + \sigma_{22}}{2} - \frac{\sigma_{11} - \sigma_{22}}{2} \cos 2\alpha - (\sin 2\alpha)\sigma_{12} 
\]

(7c)

\[
\sigma'_{12} = \sigma'_{21} = -(\cos \alpha)(\sin \alpha)\sigma_{11} + [(\cos \alpha)^2 - (\sin \alpha)^2]\sigma_{12} + (\cos \alpha)(\sin \alpha)\sigma_{22} 
\]

(8a)

\[
= -\frac{\sigma_{11} - \sigma_{22}}{2} \sin 2\alpha + (\cos 2\alpha)\sigma_{12} 
\]

(8b)
Equations (6) to (8) are identical to the stress transformation equations typically introduced in conjunction with the discussion of Mohr’s circle.

Due to their symmetries, and in order to facilitate their eventual adoption into computer programs, the (3 x 3) strain and stress matrices are further reduced to (6 x 1) vectors. The transformation law given by equation (16) is now written in the following (equivalent) form

\[ \sigma' = T^\sigma \sigma \]  

\[ \sigma^{'} = [\sigma'_{11} \sigma'_{22} \sigma'_{33} \sigma'_{12} \sigma'_{13} \sigma'_{23}]^T \]  

\[ \sigma = [\sigma_{11} \sigma_{22} \sigma_{33} \sigma_{12} \sigma_{13} \sigma_{23}]^T \]  

and

\[ T^\sigma = \begin{bmatrix} T_{11}^\sigma & T_{12}^\sigma \\ T_{21}^\sigma & T_{22}^\sigma \end{bmatrix} \]  

The sub-matrices appearing in equation (20) are defined in the following manner

\[ T_{11}^\sigma = \begin{bmatrix} (r_{11})^2 & (r_{12})^2 & (r_{13})^2 \\ (r_{21})^2 & (r_{22})^2 & (r_{23})^2 \\ (r_{31})^2 & (r_{32})^2 & (r_{33})^2 \end{bmatrix} \]  

\[ T_{12}^\sigma = 2 \begin{bmatrix} r_{11}r_{12} & r_{11}r_{13} & r_{12}r_{13} \\ r_{21}r_{22} & r_{21}r_{23} & r_{22}r_{23} \\ r_{31}r_{32} & r_{31}r_{33} & r_{32}r_{31} \end{bmatrix} \]  

\[ T_{21}^\sigma = \begin{bmatrix} r_{11}r_{21} & r_{12}r_{22} & r_{13}r_{23} \\ r_{11}r_{31} & r_{12}r_{32} & r_{13}r_{33} \\ r_{21}r_{31} & r_{22}r_{32} & r_{23}r_{33} \end{bmatrix} \]  

\[ T_{22}^\sigma = \begin{bmatrix} (r_{11}r_{22} + r_{12}r_{21}) & (r_{11}r_{23} + r_{13}r_{21}) & (r_{12}r_{23} + r_{13}r_{22}) \\ (r_{11}r_{32} + r_{12}r_{31}) & (r_{11}r_{33} + r_{13}r_{31}) & (r_{12}r_{33} + r_{13}r_{32}) \\ (r_{21}r_{32} + r_{31}r_{22}) & (r_{21}r_{33} + r_{33}r_{21}) & (r_{22}r_{33} + r_{23}r_{32}) \end{bmatrix} \]
For a two-dimensional state of stress contained in the 1-2 plane (see Figure E.2.1), \( r_{13} = r_{31} = r_{23} = r_{32} = 0, \) and \( r_{33} = 1. \) As such, equation (17) reduces to

\[
\begin{bmatrix}
\sigma'_{11} \\
\sigma'_{22} \\
\sigma'_{33} \\
\sigma'_{12}
\end{bmatrix} =
\begin{bmatrix}
(cos \alpha)^2 & (sin \alpha)^2 & 0 & 2(cos \alpha)(sin \alpha) \\
(sin \alpha)^2 & (cos \alpha)^2 & 0 & -2(cos \alpha)(sin \alpha) \\
0 & 0 & 1 & 0 \\
-(cos \alpha)(sin \alpha) & (cos \alpha)(sin \alpha) & 0 & (cos \alpha)^2 - (sin \alpha)^2
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12}
\end{bmatrix}
\]

(25)

The inverse transformation of stresses is defined by

\[\sigma = (T^\sigma)^{-1} \sigma'\]

(26)

where

\[
(T^\sigma)^{-1} =
\begin{bmatrix}
(T_{11}^\sigma)^T & 2(T_{21}^\sigma)^T \\
1/2(T_{12}^\sigma)^T & (T_{22}^\sigma)^T
\end{bmatrix}
\]

(27)

For a two-dimensional state of stress contained in the 1-2 plane (see Figure E.2.1), equation (26) reduces to

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
(cos \alpha)^2 & (sin \alpha)^2 & 0 & -2(cos \alpha)(sin \alpha) \\
(sin \alpha)^2 & (cos \alpha)^2 & 0 & 2(cos \alpha)(sin \alpha) \\
0 & 0 & 1 & 0 \\
(cos \alpha)(sin \alpha) & -(cos \alpha)(sin \alpha) & 0 & (cos \alpha)^2 - (sin \alpha)^2
\end{bmatrix}
\begin{bmatrix}
\sigma'_{11} \\
\sigma'_{22} \\
\sigma'_{33} \\
\sigma'_{12}
\end{bmatrix}
\]

(28)

If engineering measures of shearing strain are used, the transformation law for the strain vector will differ from that for the stress vector. In particular,

\[\epsilon' = T^\epsilon \epsilon\]

(29)

where

\[
\epsilon' = \begin{bmatrix}
\epsilon'_{11} & \epsilon'_{22} & \epsilon'_{33} & 2\epsilon'_{12} & 2\epsilon'_{13} & 2\epsilon'_{23}
\end{bmatrix}^T = \begin{bmatrix}
\epsilon'_{11} & \epsilon'_{22} & \epsilon'_{33} & \gamma'_{12} & \gamma'_{13} & \gamma'_{23}
\end{bmatrix}^T
\]

(30)

\[
\epsilon = \begin{bmatrix}
\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & 2\epsilon_{12} & 2\epsilon_{13} & 2\epsilon_{23}
\end{bmatrix}^T = \begin{bmatrix}
\epsilon_{11} & \epsilon_{22} & \epsilon_{33} & \gamma_{12} & \gamma_{13} & \gamma_{23}
\end{bmatrix}^T
\]

(31)

and
The inverse transformation law for strains is given by

\[ \varepsilon = \left( T^e \right)^{-1} \varepsilon' \]  

where

\[ \left( T^e \right)^{-1} = (T^\sigma)^\top \]
4. Transformation of Constitutive Matrices

If the principal axes of an anisotropic material do not coincide with the global axes (Figure 1), a transformation of the constitutive matrix $C$ must be performed. Although such a transformation is defined by equations (12) and (13), the symmetries of $\sigma$ and $\epsilon$ reduce $C$ to a $(6 \times 6)$ matrix. As such, the transformation law is simplified.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Schematic Illustration of Relation Between Principal Material Axes and Global Axes}
\end{figure}

The transformation from principal material axes to global axes is considered first. With respect to the principal material axes, the constitutive relations are

$$\sigma' = C' \epsilon'$$  \hfill (35)

Substituting equations (17) and (29) into equation (35) gives

$$T^\sigma \sigma = C' T^\epsilon \epsilon$$  \hfill (36)

It therefore follows that

$$\sigma = (T^\sigma)^{-1} C' T^\epsilon \epsilon = (T^\epsilon)^T C' T^\epsilon \epsilon$$  \hfill (37)

Thus, provided that $\sigma$ and $\epsilon$ are written in vector form, and provided that engineering measures of shearing strain are used, the transformation of $C$ is realized in the following manner

$$C = (T^\sigma)^{-1} C' (T^\sigma)^{-T} = (T^\epsilon)^T C' T^\epsilon$$  \hfill (38)
The inverse transformation is obtained in a similar manner. Beginning with

\[ \sigma = C \varepsilon \]  \quad (39)

substitute equations (26) and (33) to give

\[ \left( T^\sigma \right)^{-1} \sigma' = C \left( T^e \right)^{-1} \varepsilon' \]  \quad (40)

or

\[ \sigma' = \left( T^\sigma \right) C \left( T^e \right)^{-1} \varepsilon' = \left( T^\sigma \right) C \left( T^\sigma \right)^T \varepsilon' \]  \quad (41)

It follows that the inverse transformation of \( C \) is thus given by

\[ C' = \left( T^\sigma \right) C \left( T^\sigma \right)^T = \left( T^e \right)^{-T} C \left( T^e \right)^{-1} \]  \quad (42)

5. References
