• Problem 1

**Remark:** This problem investigates the use of higher order finite difference formulas of the same type.

Re-visit the first example problem discussed in lecture; viz.,

\[ \frac{dy}{dx} + y = e^x \quad ; \quad 0 < x < 2 \quad ; \quad y(0) = 2 \]

Using the finite difference method with *forward* differences \(O(h^2)\), obtain the approximate solution for grid spacings \((h = \Delta x)\) equal to 1.00, 0.50 and 0.25 (i.e., \(P = 2, 4\) and 8, respectively). To obtain the requisite number of equations, in each of these analyses you will need to use a *backward* difference \(O(h^2)\) at the point \(x = x_P\).

At points common to all three grids, compare the approximate solution to the exact one and to the approximate solutions obtained using forward differences \(O(h)\). Discuss your findings.
• Problem 2

Remark: This problem investigates the use of different types of finite difference formulas.

Consider a cable\(^1\) of length \(L\) that, when unloaded, is oriented in a straight line parallel to the (longitudinal) \(x\)-axis (Figure 1).

![Figure 1. Schematic illustration of a cable resting on an elastic foundation.](image)

The cable is stretched under a tension \(T(x)\) [units: \(F\)], that may vary along the length (for example, if the cable were hanging vertically in a gravitational field). The only loads on the cable, \(\bar{q}(x)\) [units: \(FL^{-1}\)], are applied transversely to the longitudinal \(x\)-axis and result in displacements \(v(x)\) [units: \(L\)] that are also transverse to this axis. The cable rests on an elastic (Winkler) foundation that is characterized by the elastic modulus \(\beta(x)\) [units: \(FL^{-2}\)].

Assuming “small” displacements,\(^2\) the equation governing the transverse displacement of the cable is:

\[
\frac{d}{dx} \left( T \frac{dv}{dx} \right) - \beta v + \bar{q} = 0 \quad ; \quad 0 < x < L
\]

Consider the case where \(T\) is constant along the cable and \(L = 1\). Also assume the special case of \(\beta/T = \bar{q}/T = 1\). the governing equation thus becomes

\[
\frac{d^2v}{dx^2} - v + 1 = 0 \quad ; \quad 0 < x < 1
\]

\(^1\)By “cable” is meant any body with one characteristic cross-sectional dimension greater than the other two, for which the extensional stiffness is much greater than the flexural stiffness. As such, strings, flexible conduits, etc. can often be considered to be “cables.”

\(^2\)This restriction has the following implications. First, it assumed that \(T(x)\) remains the same in both loaded and unloaded configurations. Second, it assumes that the slope \(\theta\) of any section of the cable relative to the \(x\)-axis is small enough that \(\sin \theta \approx \tan \theta\), or, equivalently, that \(\cos \theta \approx 1\). Thirdly, it assumes that bending stresses generated by the change in curvature are small relative to the axial stress (i.e., \(T(x)\) divided by the cross-sectional area of the cable).
Use the finite difference method with a grid spacing of \( h = \Delta x = 0.25 \). Concerning the order of approximation:

a) First obtain an approximate solution using only forward differences \( O(h) \).

b) Next obtain an approximate solution using only backward differences \( O(h) \).

c) Finally, obtain a third approximate solution using central differences \( O(h^2) \).

Compare your results to the exact solution; viz.,

\[
v(x) = 1 + \left( \frac{e^{(-1)} - 1}{e^{(1)} - e^{(-1)}} \right) e^x + \left( \frac{1 - e^{(1)}}{e^{(1)} - e^{(-1)}} \right) e^{(-x)}
\]

Does the use of central differences yield a substantial increase in accuracy of the approximate solution? If so, is this higher accuracy worth the added computational effort, if any, associated with central differences?

Remark: When solving for the unknown values of the primary dependent variables in the above problems make use of existing equation solving software. DO NOT SOLVE BY HAND!

Remark: The homework solution should be neat and easy to read.

Remark: Make sure to discuss the results obtained for both problems.