Problem 3

Remark: This problem investigates the approximation of a natural (Neumann) boundary condition using the finite difference method.

Consider the differential equation

$$\frac{d^2u}{dx^2} + u = x ; \quad 0 < x < 1$$

with boundary conditions

$$u|_{x=0} = 0$$

$$\left. \left( \frac{du}{dx} + u \right) \right|_{x=1} = 0$$

Using the finite difference method with *central* differences $O(h^2)$, obtain the approximate solution for grid spacings ($h = \Delta x$) equal to 0.500, 0.250 and 0.125 (i.e., $P = 2$, 4 and 8, respectively). Use a backward difference $O(h^2)$ to approximate the natural boundary condition at $x = 1$.

At points common to all three grids, compare the approximate solution to the exact one; viz.,

$$u(x) = x - \left( \frac{2}{\cos(1) + \sin(1)} \right) \sin x$$

Discuss your findings.
• Problem 4

Please do Exercise 4.2 in the textbook. For convenience, it is provided below:

Consider the approximate solution of the non-linear equation

\[-2u \left( \frac{d^2 u}{dx^2} \right) + \left( \frac{du}{dx} \right)^2 = 4 \quad ; \quad 0 < x < 1\]

with boundary conditions \(u(0) = 1\) and \(u(1) = 0\). Using the approximate solution

\[\hat{u} = \sum_{n=1}^{3} \alpha_n \sin(n\pi x)\]

determine the general expression for the domain and boundary residuals \(R_\Omega\) and \(R_\Gamma\), respectively.