• Problem 5

Consider the horizontal bar shown in Figure 1. The cross-sectional area of the bar is \( A = 0.050 \ m^2 \), and its length is \( L = 8.0 \ m \). The elastic modulus for the material is \( E = 74.0 \ GPa \). The bar is loaded by a distributed force of \( \bar{f}(x) = 25 \frac{2}{L^3} \ x \ kN/m \), and by a concentrated force \( \bar{Q} = 200 \ kN \) applied (in the negative \( x \)-direction) at its end.

For completeness, the boundary conditions for this problem are

\[ u|_{x=0} = 0 \quad \text{and} \quad EA \frac{du}{dx} \bigg|_{x=L} = -\bar{Q} \]

Using a suitable combination of central and backward differences \( O(h^2) \), analyze the bar using the finite difference method with grid spacings of \( h = 2.0, 1.0 \) and \( 0.50 \) meters. For each of the three cases, compare the axial displacement at mid-length (i.e., \( x = L/2 \)) and at the loaded end (i.e., \( x = L \)) of the bar. Does the approximate solution appear to be converging? Discuss your results.

![Bar diagram](image)

Figure 1. Bar subjected to distributed axial force per unit length \( (\bar{f}(x)) \) and concentrated force \( (\bar{Q}) \) at its free end.

Using the previously determined finite difference approximation for the \( P = 8 \) grid, compute approximate axial strains \( (\hat{\varepsilon}_x) \), axial stresses \( (\hat{\sigma}_x) \), and axial forces \( (\hat{P}_x) \) at the grid points. For completeness, these quantities are defined as follows:

\[
\hat{\varepsilon}_x = \frac{d\hat{u}}{dx} ; \quad \hat{\sigma}_x = E\hat{\varepsilon}_x = E\frac{d\hat{u}}{dx} ; \quad \hat{P}_x = \hat{\sigma}_x A = EA\frac{d\hat{u}}{dx}
\]

In computing the approximate axial strains \( (\hat{\varepsilon}_x) \),

• At \( x = x_0 \), use a forward difference \( O(h^2) \).
• At \( x = x_1 \) to \( x = x_7 \), use a central difference \( O(h^2) \).

• At \( x = x_8 \), use a backward difference \( O(h^2) \).

How accurate are \( \hat{\varepsilon}_x \), \( \hat{\sigma}_x \), and \( \hat{P}_x \) as compared to the exact values?

\[ \frac{d}{dx} \left( EA \frac{du}{dx} \right) + f(x) = 0 \]

Remark: The governing differential equation associated with this problem is given by equation (1.6) in the text; viz.,

where \( E \) is the elastic modulus, \( A \) is the cross-sectional area, and \( f(x) \) has units of force per unit length (i.e., \( FL^{-1} \)). For this problem, \( \bar{f}(x) = \frac{25}{2} \left( \frac{x}{L} \right)^3 \) kN/m.

The exact solution for the axial displacement is

\[
\hat{u} = \frac{1}{EA} \left[ -\frac{5}{8} \left( \frac{1}{L^3} \right) x^5 + \left( \frac{25}{8} L - \bar{Q} \right) x \right]
\]

The exact axial strain distribution is

\[
\hat{\varepsilon}_x = \frac{du}{dx} = \frac{1}{EA} \left\{ \frac{25}{8} L \left[ 1 - \left( \frac{x}{L} \right)^4 \right] - \bar{Q} \right\}
\]

The exact axial stress distribution is thus simply

\[
\hat{\sigma}_x = E \hat{\varepsilon}_x = \frac{1}{A} \left\{ \frac{25}{8} L \left[ 1 - \left( \frac{x}{L} \right)^4 \right] - \bar{Q} \right\}
\]

Finally, the exact solution for the axial force is given by

\[
\hat{P}_x = (\hat{\sigma}_x)A = \frac{25}{8} L \left[ 1 - \left( \frac{x}{L} \right)^4 \right] - \bar{Q}
\]
Problem 6

Consider the bar shown in Figure 5.7 of the textbook. Essential boundary conditions are imposed at \( x = 0 \) and \( x = L \); that is, \( u(0) = \bar{u} = 0 \) and \( u(L) = \bar{u} = 0 \). The bar is assumed to be prismatic with cross-sectional area equal to \( A \). The resulting bar is thus shown in Figure 2.

Assume an elastic modulus \( E = 2.07 \times 10^8 \) kPa; the cross-sectional area of the bar is \( A = 3.20 \times 10^{-3} \) m\(^2\); finally, the body force in the positive \( x \) direction \( b_x = 75.0 \) kN/m\(^3\) (consequently, \( f(x) = b_x A \)). The length of the bar is \( L = 3.05 \) meters. Consider the following approximate solution:

\[
\hat{u} = \sum_{m=1}^{P} \beta_m \sin \frac{m\pi x}{L}
\]

Please do the following:

1. Define the general domain residual \( R_\Omega(x) \) and the boundary residual \( R_\Gamma \) associated with the approximate solution.

2. Use the collocation method to determine values for the \( \beta_m \) for \( P = 1, 2, \) and 3. In each case, use evenly spaced collocation points.

3. Use the subdomain method to determine values for the \( \beta_m \) for \( P = 1, 2, \) and 3. In each case, use sub-domains of equal size.

4. Finally, use the Bubnov-Galerkin method to determine values for the \( \beta_m \) for \( P = 1, 2, \) and 3.

5. On a single graph for each value of \( P \), plot the different approximate solutions \( \hat{u}(x) \) and the exact solution \( u(x) \) versus distance \( (x) \) along the rod. For this problem,

\[
u(x) = \frac{f(x)}{2EA} (L - x)
\]

6. Of the four methods used, which gave the best results? Which was the easiest to use?
• Problem 7

Consider once again the bar shown in Figure 2, where the values of $E$, $A$, and $f(x) = b_x A$ are as given in Problem 6.

• Using the Rayleigh-Ritz (stationary functional) method in conjunction with the following approximate solution:

$$\hat{u} = \sum_{m=1}^{P} \beta_m \sin \frac{m\pi x}{L}$$

determine the approximate displacement $\hat{u}(x)$ in the bar for $P = 1, 2,$ and $3$. The total potential (functional) is

$$I(x, u) = \int_{0}^{L} \frac{EA}{2} \left( \frac{du}{dx} \right)^2 dx - \int_{0}^{L} f(x)u \, dx$$

• Compare the values of $\hat{u}$ with the exact solution $u(x)$ for the the primary dependent variable.

• Also compare the approximate and exact solutions for the axial stress distribution. Since the exact displacement is

$$u(x) = \frac{f(x)}{2EA} (L - x) x$$

it follows that the exact axial strain distribution is given by

$$\varepsilon_x = \frac{du}{dx} = \frac{f(x)}{2EA} (L - 2x)$$

The exact axial stress distribution is thus

$$\sigma_x = E\varepsilon_x = \frac{f(x)}{2A} (L - 2x)$$